Arab. J. Math. (2017) 6:21–29
DOI 10.1007/s40065-017-0162-8

Arabian Journal of Mathematics

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A solution of delay differential equations via Picard–Krasnoselskii hybrid iterative process

Received: 15 July 2016 / Accepted: 30 January 2017 / Published online: 16 February 2017

Abstract The purpose of this paper is to introduce Picard–Krasnoselskii hybrid iterative process which is a hybrid of Picard and Krasnoselskii iterative processes. In case of contractive nonlinear operators, our iterative scheme converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative processes in the sense of Berinde (Iterative approximation of fixed points, 2002). We support our analytic proofs with a numerical example. Using this iterative process, we also find the solution of delay differential equation.

Mathematics Subject Classification 47H09 · 47H10 · 49M05 · 54H25

الملخص

هدف هذه الورقة هو تقديم عملية بيكارد - كراسنوسلسكي تكرارية هجينة من عمليتي بيكارد تكرارية وعملية كراسنوسلسكي تكرارية. في حالة المؤثرات غير الخطية الانكماشية، يتقارب مخططنا التكراري أسرع من كل من عمليات بيكارد، ومان، وكراسنوسلسكي، وإيشكاوا التكرارية بالمعنى الموضح في بيرايند [6]. ندعم إثباتاتنا التحليلية بمثال عددي. باستخدام العملية التكرارية، نجد حلا لمعادلة تأخر تفاضلية.

1 Introduction and preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers.

Let C be a nonempty convex subset of a normed space E and $T : C \to C$ a mapping. The mapping $T : C \to C$ is said to be a contraction if

$$\|Tx - Ty\| \leq \delta \|x - y\| \quad \text{for each } x, y \in C \text{ and } \delta \in (0, 1). \tag{1.1}$$

$F(T)$ stands for the set of fixed points of T.

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The Picard or successive or repeated function iterative process [30] is defined by the sequence $\{u_n\}$ as follows:

$$\begin{cases} u_1 = u \in C, \\ u_{n+1} = Tu_n, \quad n \in \mathbb{N}. \end{cases} \tag{1.2}$$

The Mann iterative process [26] is defined by the sequence $\{v_n\}$:

$$\begin{cases} v_1 = v \in C, \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n T v_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is appropriately chosen sequence in $(0, 1)$. This is a one-step iterative process.

The Krasnoselskii iterative process [25] is defined by the sequence $\{s_n\}$ as follows:

$$\begin{cases} s_1 \in C, \\ s_{n+1} = (1-\lambda)s_n + \lambda T s_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\lambda \in (0, 1)$. This is an averaging process.

The sequence $\{z_n\}$ defined by

$$\begin{cases} z_1 = z \in C, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \quad n \in \mathbb{N} \end{cases} \tag{1.5}$$

is known as Ishikawa iterative process [22], where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriately chosen sequences in $(0, 1)$.

Most of the physical problems of applied sciences and engineering are usually formulated in the form of fixed point equations. The study of iterative processes to approximate the solution of these equations is an active area of research (see e.g., [1,23,24,28,29] and the references therein). The Picard iterative scheme is one of the simplest iteration scheme used to approximate the solution of fixed point equations involving nonlinear contractive operators. Chidume and Olaleru [13] established some interesting fixed points results using the Picard iteration process. Chidume [12] generalized and improved the results in [3]. Chidume et al. [11] established some convergence theorems for multivalued nonexpansive mappings for a Krasnoselskii-type sequence which is known to be superior to the Mann-type and Ishikawa-type iterations (see [11]). Okeke and Abbas [28] proved the convergence and almost sure T -stability of Mann-type and Ishikawa-type random iterative schemes.

Recently Khan [24] introduced the Picard–Mann hybrid iterative process. This new iterative process for one mapping case is given by the sequence $\{m_n\}$ as follows:

$$\begin{cases} m_1 = m \in C, \\ m_{n+1} = T z_n, \\ z_n = (1-\alpha_n)m_n + \alpha_n T m_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is an appropriately chosen sequence in $(0, 1)$.

Motivated by the facts above, we now introduce the Picard–Krasnoselskii hybrid iterative process defined by the sequence $\{x_n\}$:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = T y_n, \\ y_n = (1-\lambda)x_n + \lambda T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\lambda \in (0, 1)$.

Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes that converge to a certain fixed point p of a given operator T. The sequence $\{u_n\}$ is better than $\{v_n\}$ in the sense of Rhoades [31] if

$$\|u_n - p\| \le \|v_n - p\|, \quad \text{for all } n \in \mathbb{N}.$$

The following definitions are due to Berinde [6].

Definition 1.1 [6] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b, respectively. The sequence $\{a_n\}$ is said to converge faster than $\{b_n\}$ if

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0. \tag{1.8}$$

Definition 1.2 [6] Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes that converge to a certain fixed point p of a given operator T. Suppose that the error estimates

$$\|u_n - p\| \le a_n \quad \text{for all} \quad n \in \mathbb{N},$$
$$\|v_n - p\| \le b_n \quad \text{for all} \quad n \in \mathbb{N}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to p.

Several mathematicians have obtained interesting results dealing with the rate of convergence of various iterative processes (see for example, [2,5,7–9,18,20,31,32,39]). Some authors have also investigated the stability of various iterative processes for certain nonlinear operators. See, for example, Dogan and Karakaya [18], Akewe et al. [3] and the references therein.

The following lemma will be needed in the sequel.

Lemma 1.3 [34] *Let $\{s_n\}$ be a sequence of positive real numbers which satisfies:*

$$s_{n+1} \le (1 - \mu_n)s_n. \tag{1.9}$$

If $\{\mu_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} \mu_n = \infty$, then $\lim_{n\to\infty} s_n = 0$.

Interest in the study of delay differential equations stems from the fact that several models in real-life problems involves delay differential equations. For instance, delay models are common in many branches of biological modeling (see [19]). They have been used for describing several aspects of infectious disease dynamics: primary infection [14], drug therapy [27] and immune response [16], among others. These models have also appeared in the study of chemostat models [40], circadian rhythms [33], epidemiology [17], the respiratory system [37], tumor growth [38] and neural networks [10]. Statistical analysis of ecological data (see e.g., [35,36]) has shown that there is evidence of delay effects in the population dynamics of many species.

The aim of this paper is to introduce the Picard–Krasnoselskii hybrid iterative process and to show that this new iterative process is faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative processes in the sense of Berinde [6]. Finally, we show that our iterative process can be used to find the solution of delay differential equations.

2 Rate of convergence

In this section, we prove that the Picard–Krasnoselskii hybrid iterative process (1.7) converges at a rate faster than all of Picard iterative process (1.2), Mann iterative process (1.3), Krasnoselskii iterative process (1.4) and Ishikawa iterative process (1.5).

Proposition 2.1 *Let C be a nonempty closed convex subset of a normed space E and $T : C \to C$ a contraction mapping. Suppose that each of the iterative processes (1.2), (1.3), (1.4), (1.5) and (1.7) converges to the same fixed point p of T, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ such that $0 < \alpha \le \lambda, \alpha_n, \beta_n < 1$ for all $n \in \mathbb{N}$ and for some α. Then the Picard–Krasnoselskii hybrid iterative process (1.7) converges faster than all the other four processes.*

Proof Suppose that p is the fixed point of the operator T. Using (1.1) and the Picard iterative process (1.2), we have

$$\begin{aligned}\|u_{n+1} - p\| &= \|Tu_n - p\| \\ &\le \delta\|u_n - p\| \\ &\ \ \vdots \\ &\le \delta^n\|u_1 - p\|.\end{aligned} \tag{2.1}$$

Let

$$a_n = \delta^n \|u_1 - p\|. \tag{2.2}$$

Using (1.1) and the Mann iterative process (1.3), we obtain that

$$\begin{aligned}
\|v_{n+1} - p\| &= \|(1-\alpha_n)(v_n - p) + \alpha_n(Tv_n - p)\| \\
&\le (1-\alpha_n)\|v_n - p\| + \alpha_n\delta\|v_n - p\| \\
&= (1-(1-\delta)\alpha_n)\|v_n - p\| \\
&\le (1-(1-\delta)\alpha)\|v_n - p\| \\
&\vdots \\
&\le (1-(1-\delta)\alpha)^n\|v_1 - p\|.
\end{aligned} \tag{2.3}$$

Set

$$b_n = (1-(1-\delta)\alpha)^n\|v_1 - p\|. \tag{2.4}$$

By (1.1) and the Krasnoselskii iterative process (1.4), we get

$$\begin{aligned}
\|s_{n+1} - p\| &= \|(1-\lambda)(s_n - p) + \lambda(Ts_n - p)\| \\
&\le (1-\lambda)\|s_n - p\| + \lambda\delta\|s_n - p\| \\
&= (1-(1-\delta)\lambda)\|s_n - p\| \\
&\le (1-(1-\delta)\alpha)\|s_n - p\| \\
&\vdots \\
&\le (1-(1-\delta)\alpha)^n\|s_n - p\|.
\end{aligned} \tag{2.5}$$

Put

$$c_n = (1-(1-\delta)\alpha)^n\|s_1 - p\|. \tag{2.6}$$

From (1.1) and the Ishikawa iterative process (1.5), it follows that

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)(z_n - p) + \beta_n(Tz_n - p)\| \\
&\le (1-\beta_n)\|z_n - p\| + \beta_n\delta\|z_n - p\|.
\end{aligned} \tag{2.7}$$

From (1.5), (1.1) and (2.7), we obtain that

$$\begin{aligned}
\|z_{n+1} - p\| &= \|(1-\alpha_n)(z_n - p) + \alpha_n(Ty_n - p)\| \\
&\le (1-\alpha_n)\|z_n - p\| + \alpha_n\delta\|y_n - p\| \\
&\le (1-\alpha_n)\|z_n - p\| + \alpha_n\delta[(1-\beta_n)\|z_n - p\| + \beta_n\delta\|z_n - p\|] \\
&= (1-\alpha_n)\|z_n - p\| + \alpha_n\delta(1-\beta_n)\|z_n - p\| + \alpha_n\beta_n\delta^2\|z_n - p\| \\
&\le (1-\alpha_n)\|z_n - p\| + \alpha_n\delta\|z_n - p\| \\
&= (1-(1-\delta)\alpha_n)\|z_n - p\| \\
&\le (1-(1-\delta)\alpha)\|z_n - p\| \\
&\vdots \\
&\le (1-(1-\delta)\alpha)^n\|z_1 - p\|.
\end{aligned} \tag{2.8}$$

Let

$$e_n = (1-(1-\delta)\alpha)^n\|z_1 - p\|. \tag{2.9}$$

Using (1.1) and the Picard–Krasnoselskii hybrid iterative process (1.7), we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|Ty_n - p\| \\
&\leq \delta\|y_n - p\| \\
&\leq \delta\|(1-\lambda)(x_n - p) + \lambda(Tx_n - p)\| \\
&\leq \delta[(1-\lambda)\|x_n - p\| + \lambda\delta\|x_n - p\|] \\
&= \delta(1-(1-\delta)\lambda)\|x_n - p\| \\
&\leq \delta(1-(1-\delta)\alpha)\|x_n - p\| \\
&\ \ \vdots \\
&\leq [\delta(1-(1-\delta)\alpha)]^n\|x_1 - p\|. \qquad (2.10)
\end{aligned}$$

Set:

$$h_n = [\delta(1-(1-\delta)\alpha)]^n\|x_1 - p\|. \qquad (2.11)$$

We now compute the rate of convergence of our iterative process (1.7) as follows:

(i) Note that

$$\frac{h_n}{a_n} = \frac{[\delta(1-(1-\delta)\alpha)]^n\|x_1 - p\|}{\delta^n\|u_1 - p\|} = [(1-(1-\delta)\alpha)]^n\frac{\|x_1 - p\|}{\|u_1 - p\|} \to 0 \text{ as } n \to \infty. \qquad (2.12)$$

Thus, $\{x_n\}$ converges faster than $\{u_n\}$ to p. That is, the Picard–Krasnoselskii hybrid iterative process (1.7) converges faster than the Picard iterative process (1.2) to p.

(ii) Similarly,

$$\frac{h_n}{b_n} = \frac{[\delta(1-(1-\delta)\alpha)]^n\|x_1 - p\|}{(1-(1-\delta)\alpha)^n\|v_1 - p\|} = \delta^n\frac{\|x_1 - p\|}{\|v_1 - p\|} \to 0 \text{ as } n \to \infty. \qquad (2.13)$$

Hence, $\{x_n\}$ converges faster than $\{v_n\}$ to p.

(iii) Clearly, $\frac{h_n}{c_n} = \delta^n\frac{\|x_1-p\|}{\|s_1-p\|} \to 0$ as $n \to \infty$. Hence, $\{x_n\}$ converges faster than $\{s_n\}$ to p.

(iv) Finally, $\frac{h_n}{e_n} = \delta^n\frac{\|x_1-p\|}{\|z_1-p\|} \to 0$ as $n \to \infty$. Hence, $\{x_n\}$ converges faster than $\{z_n\}$ to p. This completes the proof of Proposition 2.1.

□

Next, we give a numerical example to support Proposition 2.1.

Example 2.2 Let $C = [1, 10] \subseteq X = \mathbb{R}$ and $T : C \to C$ be an operator defined by $Tx = \sqrt[3]{2x+4}$ for all $x \in C$. Choose $\alpha_n = \beta_n = \lambda = \frac{1}{2}$ for each $n \in \mathbb{N}$ with the initial value $x_1 = 5$. Clearly, T is a contraction mapping with contractive constant $\delta = \frac{1}{\sqrt[3]{4}}$ and $F(T) = \{2\}$. Tables 1 and 2 show that our iterative process (1.7) converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative processes.

Remark 2.3 Clearly, from Tables 1 and 2, we conclude that our newly introduced iterative process (1.7) converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative processes, since it converges to the fixed point $p = 2$ of T at step 14, while the Picard, Mann, Krasnoselskii and Ishikawa iterative processes fails to converge to p at step 14.

3 Application to delay differential equations

We now employ our iterative process (1.7) to find the solution of delay differential equations.

Let the space $C([a, b])$ of all continuous real-valued functions on a closed interval $[a, b]$ be endowed with the Chebyshev norm $\|x - y\|_\infty = \max_{t\in[a,b]}|x(t) - y(t)|$. It is known that $(C([a, b]), \|.\|_\infty)$ is a Banach space ([21]).

In this section, we consider the following delay differential equation.

$$x'(t) = f(t, x(t), x(t-\tau)), \quad t \in [t_0, b], \qquad (3.1)$$

Table 1 Comparison of the speed of convergence among various iterative processes

Step	Picard–Krasnoselskii	Picard	Krasnoselskii	Ishikawa
0	5.0000000000000	5.0000000000000	5.0000000000000	5.0000000000000
1	2.2512843540734	2.4101422641752	3.7050711320876	3.6256421770367
2	2.0240689690982	2.0661453253859	2.9781777430805	2.8852207823505
3	2.0023366393861	2.0109640063486	2.5647367768450	2.4835391284559
4	2.0002271411589	2.0018256673537	2.3273739038994	2.2646218411812
5	2.0000220828647	2.0003042316115	2.1902559473071	2.1449743461988
6	2.0000021469423	2.0000507039831	2.1107377043938	2.0794736882823
7	2.0000002087305	2.0000084506281	2.0645131216910	2.0435817310182
8	2.0000000202932	2.0000014084370	2.0376040081566	2.0239038614161
9	2.0000000019730	2.0000002347395	2.0219259025214	2.0131122475407
10	2.0000000001918	2.0000000391232	2.0127867814254	2.0071930196537
11	2.0000000000186	2.0000000065205	2.0074578224146	2.0039460184220
12	2.0000000000018	2.0000000010868	2.0043500105642	2.0021647837650
13	2.0000000000002	2.0000000001811	2.0025373748349	2.0011876106441
14	2.0000000000000	2.0000000000302	2.0014800906259	2.0006515322465
⋮	⋮	⋮	⋮	⋮

Table 2 Comparison of the speed of convergence among various iterative processes

Step	Picard–Krasnoselskii	Mann
0	5.0000000000000	5.0000000000000
1	2.2512843540734	3.7050711320876
2	2.0240689690982	2.9781777430805
3	2.0023366393861	2.5647367768450
4	2.0002271411589	2.3273739038994
5	2.0000220828647	2.1902559473071
6	2.0000021469423	2.1107377043938
7	2.0000002087305	2.0645131216910
8	2.0000000202932	2.0376040081566
9	2.0000000019730	2.0219259025214
10	2.0000000001918	2.0127867814254
11	2.0000000000186	2.0074578224146
12	2.0000000000018	2.0043500105642
13	2.0000000000002	2.0025373748349
14	2.0000000000000	2.0014800906259
⋮	⋮	⋮

with initial condition

$$x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0]. \tag{3.2}$$

By the solution of above problem, we mean a function $x \in C([t_0 - \tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$ satisfying (3.1), (3.2).

Assume that the following conditions are satisfied.

(C_1) $t_0, b \in \mathbb{R}$, $\tau > 0$;
(C_2) $f \in C([t_0, b] \times \mathbb{R}^2, \mathbb{R})$;
(C_3) $\varphi \in C([t_0 - \tau, b], \mathbb{R})$;
(C_4) there exist $L_f > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f \sum_{i=1}^{2} |u_i - v_i|, \quad \forall u_i, v_i \in \mathbb{R}, \ i = 1, 2, \ t \in [t_0, b]; \tag{3.3}$$

(C_5) $2L_f(b - t_0) < 1$.

Now, we reformulate Problem (3.1), (3.2) by following integral equation:

$$x(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))\mathrm{d}s, & t \in [t_0, b]. \end{cases} \tag{3.4}$$

Coman et al. [15] established the following results.

Theorem 3.1 *Assume that conditions* (C_1)–(C_5) *are satisfied. Then Problem* (3.1), (3.2) *has a unique solution, say* p, *in* $C([t_0 - \tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$ *and*

$$p = \lim_{n\to\infty} T^n(x) \quad \text{for any } x \in C([t_0 - \tau, b], \mathbb{R}). \tag{3.5}$$

Next, we prove the following result using our iterative process (1.7).

Theorem 3.2 *Assume that conditions* (C_1)–(C_5) *are satisfied. Then Problem* (3.1), (3.2) *has a unique solution* p *(say), in* $C([t_0 - \tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$ *and the Picard–Krasnoselskii hybrid iterative process* (1.7) *converges to* p.

Proof Let $\{x_n\}$ be an iterative sequence generated by the Picard–Krasnoselskii hybrid iterative process (1.7) for an operator defined by

$$Tx(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))\mathrm{d}s, & t \in [t_0, b]. \end{cases} \tag{3.6}$$

Let p be a fixed point of T. We now prove that $x_n \to p$ as $n \to \infty$. It is easy to see that $x_n \to p$ for each $t \in [t_0 - \tau, t_0]$. Now, for each $t \in [t_0, b]$ we have

$$\begin{aligned}
\|y_n - p\|_\infty &= \|(1-\lambda)x_n + \lambda T x_n - p\|_\infty \\
&\le (1-\lambda)\|x_n - p\|_\infty + \lambda \|Tx_n - Tp\|_\infty \\
&= (1-\lambda)\|x_n - p\|_\infty + \lambda \max_{t\in[t_0-\tau,b]} |Tx_n(t) - Tp(t)| \\
&= (1-\lambda)\|x_n - p\|_\infty + \lambda \max_{t\in[t_0-\tau,b]} |\varphi(t_0) \\
&\quad + \int_{t_0}^{t} f(s, x_n(s), x_n(s-\tau))\mathrm{d}s - \varphi(t_0) - \int_{t_0}^{t} f(s, p(s), p(s-\tau))\mathrm{d}s| \\
&= (1-\lambda)\|x_n - p\|_\infty + \lambda \max_{t\in[t_0-\tau,b]} |\int_{t_0}^{t} f(s, x_n(s), x_n(s-\tau))\mathrm{d}s \qquad (3.7) \\
&\quad - \int_{t_0}^{t} f(s, p(s), p(s-\tau))\mathrm{d}s| \\
&\le (1-\lambda)\|x_n - p\|_\infty + \lambda \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, x_n(s), x_n(s-\tau)) \qquad (3.8) \\
&\quad - f(s, p(s), p(s-\tau))|\mathrm{d}s \\
&\le (1-\lambda)\|x_n - p\|_\infty + \lambda \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|x_n(s) - p(s)| \\
&\quad + |x_n(s-\tau) - p(s-\tau)|)\mathrm{d}s \\
&\le (1-\lambda)\|x_n - p\|_\infty + \lambda \int_{t_0}^{t} L_f(\max_{s\in[t_0-\tau,b]} |x_n(s) - p(s)| \\
&\quad + \max_{s\in[t_0-\tau,b]} |x_n(s-\tau) - p(s-\tau)|)\mathrm{d}s \\
&\le (1-\lambda)\|x_n - p\|_\infty + \lambda \int_{t_0}^{t} L_f(\|x_n - p\|_\infty + \|x_n - p\|_\infty)\mathrm{d}s \\
&\le (1-\lambda)\|x_n - p\|_\infty + 2\lambda L_f(t - t_0)\|x_n - p\|_\infty \\
&\le [1 - (1 - 2L_f(b - t_0))\lambda]\|x_n - p\|_\infty. \qquad (3.9)
\end{aligned}$$

Using (1.7) and (3.7), we obtain that

$$\begin{aligned}\|x_{n+1} - p\|_\infty &= \|Ty_n - Tp\|_\infty \\ &= \max_{t\in[t_0-\tau,b]} \left| \int_{t_0}^{t} [f(s, y_n(s), y_n(s-\tau)) - f(s, p(s), p(s-\tau))]\mathrm{d}s \right| \\ &\le \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, y_n(s), y_n(s-\tau)) - f(s, p(s), p(s-\tau))|\,\mathrm{d}s \\ &\le \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|y_n(s) - p(s)| + |y_n(s-\tau) - p(s-\tau)|)\mathrm{d}s \\ &\le 2L_f(b-t_0)\|y_n - p\|_\infty. \end{aligned} \tag{3.10}$$

It follows from (3.7) and (3.10) that

$$\|x_{n+1} - p\|_\infty \le 2L_f(b-t_0)[1-(1-2L_f(b-t_0))\lambda]\|x_n - p\|_\infty. \tag{3.11}$$

Using condition (C_5) in (3.11), we have:

$$\|x_{n+1} - p\|_\infty \le (1-(1-2L_f(b-t_0))\lambda)\|x_n - p\|_\infty. \tag{3.12}$$

Note that $(1-(1-2L_f(b-t_0))\lambda) = \mu_n < 1$ and $\|x_n - p\|_\infty = s_n$. Thus all the conditions of Lemma 1.3 are satisfied. Hence, $\lim_{n\to\infty} \|x_n - p\|_\infty = 0$. This completes the proof of Theorem 3.2. □

Remark 3.3 Theorem 3.2 generalizes and improves several known results in literature including the results of Coman et al. [15].

Acknowledgements The authors wish to thank the anonymous referees for their useful comments and suggestions which led to the improvement of the paper.

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